

ON GENERALIZED MAX-LINEAR MODELS

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ABSTRACT. We propose a way how to generate a max-stable process in $C[0, 1]$ from a max-stable random vector in \mathbb{R}^d by generalizing the max-linear model established by Wang and Stoev (2011). It turns out that if the random vector follows some finite dimensional distribution of some initial max-stable process, the approximating processes converge uniformly to the original process and the pointwise mean squared error can be represented in a closed form. The obtained results carry over to the case of generalized Pareto processes. The introduced method enables the reconstruction of the initial process only from a finite set of observation points and, thus, reasonable prediction of max-stable processes gets possible.

1. INTRODUCTION AND PRELIMINARIES

1.1. Introduction. A max-stable process (MSP) $\xi = (\xi_t)_{t \in K}$ with sample paths in $C(K) := \{f \in \mathbb{R}^K : f \text{ is continuous}\}$ with a compact set $K \subset \mathbb{R}$ has the characteristic property that there are for every $n \in \mathbb{N}$ continuous functions $a_n > 0, b_n \in C(K)$ such that

$$\max_{1 \leq i \leq n} (\xi^{(i)} + b_n)/a_n = \left(\max_{1 \leq i \leq n} (\xi_t(i) + b_n(t))/a_n(t) \right)_{t \in K} =_d \xi,$$

where the $\xi^{(i)}$ are independent copies of ξ and “ $=_d$ ” denotes equality in distribution. It is well known (e.g. de Haan and Ferreira (2006)), that MSP are the only possible limit processes of linearly standardized maxima of independent and identically distributed processes. This is in complete accordance to the well-established finite dimensional case of extreme value analysis, see Falk et al. (2010) and again de Haan and Ferreira (2006), among others.

The theory on continuous max-stable processes is essentially based on the early works of de Haan (1984), Giné et al. (1990) and de Haan and Lin (2001), followed by recent findings with the focus on different aspects within the theory, e.g. Hult

2010 *Mathematics Subject Classification.* Primary 60G70.

Key words and phrases. Multivariate extreme value distribution • multivariate generalized Pareto distribution • max-stable process • generalized Pareto process • D -norm • max-linear model • prediction of max-stable and generalized Pareto process.

and Lindskog (2005, 2006), Stoev and Taqqu (2005), Davis and Mikosch (2008), Kabluchko (2009), Wang and Stoev (2010), Aulbach et al. (2012).

Moreover, the class of excursion stable generalized Pareto processes, which is closely related to the class of max-stable processes, was examined in Buishand et al. (2008) and Aulbach et al. (2012), the most recent advances on that issue can be found in Ferreira and de Haan (2012).

There is a very crucial problem of the theory on stochastic processes concerning its relevance in practice: as (continuous) processes as a whole cannot be measured exactly, the question arises how to construct those processes (with some characteristic stochastic behavior such as max-stability) from a finite set of observations. As an example one can think of data from a finite set of measuring stations measuring the sea level along a coast and one is interested in predicting the sea level between those measuring stations.

This is an issue of conditional sampling and in the case of max-stable processes (or fields, if the domain has more than one dimension) there are (partial) answers on the arising questions in Wang and Stoev (2011) and Dombry et al. (2012).

In this paper, we pick up the so-called “max-linear model” introduced in Wang and Stoev (2011): for appropriately chosen nonnegative continuous functions g_0, \dots, g_d one obtains an MSP $\boldsymbol{\eta} = (\eta_t)_{t \in [0,1]}$ by setting

$$\eta_s = \max_{j=0, \dots, d} \frac{X_j}{g_j(s)}, \quad s \in [0, 1],$$

where $\mathbf{X} = (X_0, \dots, X_d)$ is a max-stable rv with independent components. The obvious restriction of this model is the required independence of the margins of \mathbf{X} which results in the fact that $\boldsymbol{\eta}$ has always a discrete spectral measure (the latter is in our setup described below essentially the distribution of the generator process \mathbf{Z} ; see Aulbach et al. (2012) for details).

In Section 2, we generalize this model by allowing arbitrary dependence structures of the margins of the max-stable rv \mathbf{X} . This leads immediately to the main issue of this paper, the reconstruction of a max-stable process which is observed only through a finite set of indices: it is shown in Section 3 that if the random vector is some finite dimensional projection of some initial MSP, the processes resulting from the construction in Section 2 converge uniformly to the original process as the grid of indices gets finer. Moreover, the mean squared error between the predictive and the original process is computed at a fixed index, which is useful for practical purposes.

It is also possible to predict generalized Pareto processes with the same techniques as for MSP which is the content of Section 4.

To begin with, the following subsection recalls some basic theory needed in what follows and introduces some notation. For the ease of notation we choose $K = [0, 1]$ as domain of the processes, being aware of the fact that all results are valid for an arbitrary compact set $K \subset \mathbb{R}$ as well. The extension of the results to more general domains (in particular higher dimensions of the domain) is not immediately obvious and subject of current research.

1.2. Preliminaries. We call a random vector (rv) $\mathbf{X} = (X_0, \dots, X_d)$ standard max-stable, if it is max-stable and each component follows the standard negative exponential distribution, i.e., $P(X_i \leq x) = \exp(x)$, $x \leq 0$, $i = 0, \dots, d$. It is well-known (e.g. de Haan and Resnick (1977), Pickands (1981), Falk et al. (2010, Sections 4.2, 4.3)) that \mathbf{X} is standard max-stable if and only if (iff) there exists a rv $\mathbf{Z} = (Z_0, \dots, Z_d)$ with $Z_i \in [0, c]$ almost surely and $E(Z_i) = 1$, $i = 0, \dots, d$, for some number $c \geq 1$, such that

$$P(\mathbf{X} \leq \mathbf{x}) = \exp(-\|\mathbf{x}\|_D) := \exp\left(-E\left(\max_{0 \leq i \leq d} (|x_i| Z_i)\right)\right), \quad \mathbf{x} \leq \mathbf{0} \in \mathbb{R}^{d+1}.$$

Note that $\|\cdot\|_D$ defines a norm on \mathbb{R}^{d+1} , called D -norm, with generator \mathbf{Z} . The D means dependence: We have independence of the margins of \mathbf{X} iff $\|\cdot\|_D$ equals the norm $\|\mathbf{x}\|_1 = \sum_{i=0}^d |x_i|$, which is generated by (Z_0, \dots, Z_d) being a random permutation of the vector $(d+1, 0, \dots, 0)$. We have complete dependence of the margins of \mathbf{X} iff $\|\cdot\|_D$ is the maximum-norm $\|\mathbf{x}\|_\infty = \max_{0 \leq i \leq d} |x_i|$, which is generated by the constant vector $(Z_0, \dots, Z_d) = (1, \dots, 1)$. We refer to Falk et al. (2010, Section 4.4) for further details of D -norms.

We call a stochastic process $\boldsymbol{\eta}$ with sample paths in $\bar{C}^-[0, 1] := \{f \in C[0, 1] : f \leq 0\}$ a standard max-stable process (SMSP), if it is a max-stable process with standard negative exponential univariate margins.

Denote by $E[0, 1]$ the set of those bounded functions $f : [0, 1] \rightarrow \mathbb{R}$ that have only a finite number of discontinuities, and let $\bar{E}^-[0, 1]$ be the subset of those functions in $E[0, 1]$, which attain only non-positive values.

It is known from Giné et al. (1990) and Aulbach et al. (2012, Lemma 2) that a stochastic process $\boldsymbol{\eta}$ in $C[0, 1]$ is an SMSP iff there exists a stochastic process $\mathbf{Z} = (Z_t)_{t \in [0, 1]}$ with sample paths in $\bar{C}^+[0, 1] := \{f \in C[0, 1] : f \geq 0\}$ with $Z_t \leq c$ a.s. and $E(Z_t) = 1$, $t \in [0, 1]$, for some constant $c \geq 1$, such that

$$P(\boldsymbol{\eta} \leq f) = \exp(-\|f\|_D) := \exp\left(-E\left(\sup_{t \in [0, 1]} (|f(t)| Z_t)\right)\right), \quad f \in \bar{E}^-[0, 1].$$

The condition $P(\sup_{t \in [0, 1]} Z_t \leq c) = 1$ on the generator process \mathbf{Z} can be weakened to $E\left(\sup_{t \in [0, 1]} Z_t\right) < \infty$, see de Haan and Ferreira (2006, Corollary 9.4.5).

Note that $\|\cdot\|_D$ defines a norm again, this time on the space $E[0, 1]$. It is also called D -norm with generator process \mathbf{Z} , and we have

$$\|f\|_\infty \leq \|f\|_D \leq \varepsilon_D \|f\|_\infty, \quad f \in E[0, 1],$$

where $\|f\|_\infty = \sup_{t \in [0, 1]} |f(t)|$ and $\varepsilon_D = \|1\|_D = E(\|\mathbf{Z}\|_\infty)$ is the extremal coefficient, cf. Smith (1990). This implies $\|\cdot\|_D = \|\cdot\|_\infty$ iff $\varepsilon_D = 1$, cf. Aulbach et al. (2012). Moreover, the preceding inequality shows that each D -norm on the space $E[0, 1]$ is equivalent to the sup-norm $\|\cdot\|_\infty$, which is itself a D -norm by putting $Z_t = 1$, $t \in [0, 1]$.

We conclude this section by introducing generalized Pareto processes as considered in Section 4. For the purpose of this paper, the following definition is sufficient: we call a stochastic process \mathbf{V} in $\bar{C}^-[0, 1]$ a standard generalized Pareto process (SGPP), if there exists a D -norm $\|\cdot\|_D$ on $E[0, 1]$ and some $c > 0$, such that $P(\mathbf{V} \leq f) = 1 - \|f\|_D$ for all $f \in \bar{E}^-[0, 1]$ with $\|f\|_\infty \leq c$. For a detailed examination of GPP we refer to Ferreira and de Haan (2012).

2. THE GENERALIZED MAX-LINEAR MODEL

Let $\mathbf{X} = (X_0, \dots, X_d)$ be a standard max-stable rv with pertaining D -norm $\|\cdot\|_{D_{0, \dots, d}}$ on \mathbb{R}^{d+1} generated by $\mathbf{Z} = (Z_0, \dots, Z_d)$, $d \in \mathbb{N}$, i. e.

$$P(\mathbf{X} \leq \mathbf{x}) = \exp\left(-\|\mathbf{x}\|_{D_{0, \dots, d}}\right) = \exp\left(-E\left(\max_{i=0, \dots, d} |x_i| Z_i\right)\right),$$

$\mathbf{x} = (x_0, \dots, x_d) \leq \mathbf{0}$. Choose deterministic functions $g_0, \dots, g_d \in \bar{C}^+[0, 1]$ with the property

$$(1) \quad \|(g_0(t), \dots, g_d(t))\|_{D_{0, \dots, d}} = 1, \quad t \in [0, 1].$$

For instance, in case of independent margins of \mathbf{X} , we have $\|\cdot\|_{D_{0, \dots, d}} = \|\cdot\|_1$, and condition (1) becomes

$$\sum_{i=0}^d g_i(t) = 1, \quad t \in [0, 1],$$

i. e. $g_i(t)$, $i = 0, \dots, d$, defines a probability distribution on the set $\{0, \dots, d\}$ for each $t \in [0, 1]$. In this case, an example is given by the binomial distribution

$$g_i(t) := \binom{d}{i} t^i (1-t)^{d-i}, \quad i = 0, \dots, d, \quad t \in [0, 1].$$

Put now for $t \in [0, 1]$

$$(2) \quad \eta_t := \max_{i=0, \dots, d} \frac{X_i}{g_i(t)}.$$

The model (2) is called generalized max-linear model. It defines an SMSP as the next lemma shows:

Lemma 2.1. *The stochastic process $\boldsymbol{\eta} = (\eta_t)_{t \in [0,1]}$ in (2) defines an SMSP with generator process $\hat{\mathbf{Z}} = (\hat{Z}_t)_{t \in [0,1]}$ given by*

$$\hat{Z}_t = \max_{i=0,\dots,d} (g_i(t)Z_i), \quad t \in [0,1].$$

Proof. At first we verify that the process $\hat{\mathbf{Z}}$ is indeed a generator process. It is obvious that the sample paths of $\hat{\mathbf{Z}}$ are in $\bar{C}^+[0,1]$. Furthermore, we have by construction for each $t \in [0,1]$

$$E(\hat{Z}_t) = \|(g_0(t), \dots, g_d(t))\|_{D_{0,\dots,d}} = 1.$$

As $\|\cdot\|_\infty \leq \|\cdot\|_D$ for an arbitrary D -norm, we have $\|(g_0(t), \dots, g_d(t))\|_\infty \leq 1$, $t \in [0,1]$, and, thus, $\hat{Z}_t \leq \max_{i=0,\dots,d} Z_i$, $t \in [0,1]$.

In addition, we have for $f \in \bar{E}^-[0,1]$

$$\begin{aligned} P(\boldsymbol{\eta} \leq f) &= P(X_i \leq g_i(t)f(t), \quad i = 0, \dots, d, \quad t \in [0,1]) \\ &= P\left(X_i \leq \inf_{t \in [0,1]} (g_i(t)f(t)), \quad i = 0, \dots, d\right) \\ &= P\left(X_i \leq - \sup_{t \in [0,1]} (g_i(t)|f(t)|), \quad i = 0, \dots, d\right) \\ &= \exp\left(-\left\|\left(\sup_{t \in [0,1]} (g_0(t)|f(t)|), \dots, \sup_{t \in [0,1]} (g_d(t)|f(t)|)\right)\right\|_{D_{0,\dots,d}}\right) \\ &= \exp\left(-E\left(\max_{i=0,\dots,d} \left(\sup_{t \in [0,1]} (g_i(t)|f(t)|) Z_i\right)\right)\right) \\ &= \exp\left(-E\left(\sup_{t \in [0,1]} \left(|f(t)| \max_{i=0,\dots,d} (g_i(t)Z_i)\right)\right)\right) \\ &= \exp\left(-E\left(\sup_{t \in [0,1]} (|f(t)| \hat{Z}_t)\right)\right) \end{aligned}$$

which completes the proof. \square

REMARK 2.2. Condition (1) ensures that the univariate margins η_t , $t \in [0,1]$, of the process $\boldsymbol{\eta}$ in model (2) follow the standard negative exponential distribution $P(\eta_t \leq x) = \exp(x)$, $x \leq 0$. If we drop this condition, we still obtain a max-stable process: Take for $n \in \mathbb{N}$ i.i.d. copies $\boldsymbol{\eta}^{(1)}, \dots, \boldsymbol{\eta}^{(n)}$ of $\boldsymbol{\eta}$. We have for $f \in \bar{E}^-[0,1]$

$$\begin{aligned} P\left(n \max_{1 \leq k \leq n} \boldsymbol{\eta}^{(k)} \leq f\right) &= P\left(\boldsymbol{\eta} \leq \frac{f}{n}\right)^n \\ &= P\left(X_i \leq \inf_{t \in [0,1]} \left(\frac{g_i(t)f(t)}{n}\right), \quad i = 0, \dots, d\right)^n \end{aligned}$$

$$\begin{aligned}
&= \exp \left(- \left\| \left(\sup_{t \in [0,1]} \left(\frac{g_0 |f(t)|}{n} \right), \dots, \sup_{t \in [0,1]} \left(\frac{g_d(t) |f(t)|}{n} \right) \right) \right\|_{D_{0,\dots,d}} \right)^n \\
&= \exp \left(- \left\| \left(\sup_{t \in [0,1]} (g_0 |f(t)|), \dots, \sup_{t \in [0,1]} (g_d(t) |f(t)|) \right) \right\|_{D_{0,\dots,d}} \right) \\
&= P(\boldsymbol{\eta} \leq f).
\end{aligned}$$

The univariate margins of $\boldsymbol{\eta}$ are now given by

$$(3) \quad P(\eta_t \leq x) = \exp \left(\|(g_0(t), \dots, g_d(t))\|_{D_{0,\dots,d}} \cdot x \right), \quad x \leq 0, \quad t \in [0, 1].$$

Note that the above calculations also give an alternative proof of Lemma 2.1, except that we do not obtain the generator process of $\boldsymbol{\eta}$ with this approach.

In model (2) we have not made any further assumptions on the D -norm $\|\cdot\|_{D_{0,\dots,d}}$, that is, on the dependence structure of the random variables X_0, \dots, X_d . The special case $\|\cdot\|_{D_{0,\dots,d}} = \|\cdot\|_1$ characterizes the case where X_0, \dots, X_d are independent. This is the regular max-linear model, cf. Wang and Stoev (2011).

On the contrary, $\|\cdot\|_{D_{0,\dots,d}} = \|\cdot\|_\infty$ provides the case of complete dependence $X_0 = \dots = X_d$ a.s. with the constant generator $Z_0 = \dots = Z_d = 1$. Thus, condition (1) becomes $\max_{i=0,\dots,d} g_i(t) = 1, t \in [0, 1]$, and therefore

$$\hat{Z}_t = \max_{i=0,\dots,d} (g_i(t) Z_i) = \max_{i=0,\dots,d} g_i(t) = 1, \quad t \in [0, 1].$$

3. RECONSTRUCTION OF SMSP

The preceding approach offers a way to reconstruct an SMSP in an appropriate way. Let $\boldsymbol{\eta} = (\eta_t)_{t \in [0,1]}$ be an SMSP with generator process $\boldsymbol{Z} = (Z_t)_{t \in [0,1]}$ and D -norm $\|\cdot\|_D$. Choose a grid $0 =: s_0 < s_1 < \dots < s_{d-1} < s_d := 1$ of points within $[0, 1]$. Then $(\eta_{s_0}, \dots, \eta_{s_d})$ is a standard max-stable rv in \mathbb{R}^{d+1} with pertaining D -norm $\|\cdot\|_{D_{0,\dots,d}}$ generated by $(Z_{s_0}, \dots, Z_{s_d})$.

The aim of this section is to define some SMSP $\hat{\boldsymbol{\eta}} = (\hat{\eta}_t)_{t \in [0,1]}$ for which $\hat{\eta}_{s_i} = \eta_{s_i}$, $i = 0, \dots, d$, holds, i.e. $\hat{\boldsymbol{\eta}}$ interpolates the finite dimensional projections $(\eta_{s_0}, \dots, \eta_{s_d})$ of the original SMSP $\boldsymbol{\eta}$ in an appropriate way. This will be done by means of a special case of the generalized max-linear model and we show that this way of “reconstructing” the original MSP $\boldsymbol{\eta}$ is reasonable as the pointwise mean squared error $\text{MSE} \left(\hat{\eta}_t^{(d)} \right) := E \left(\left(\eta_t - \hat{\eta}_t^{(d)} \right)^2 \right)$ diminishes for all $t \in [0, 1]$ as d increases. Indeed, there is additionally uniform convergence of the “predictive” processes and the corresponding generator processes to the original ones.

3.1. Uniform convergence of the discretized versions. As we have shown in Lemma 2.1, the stochastic process $\hat{\eta} = (\hat{\eta}_t)_{t \in [0,1]}$ in (2) defines an SMSP with generator process $\hat{\mathbf{Z}} = (\hat{Z}_t)_{t \in [0,1]}$, given by

$$\hat{Z}_t = \max_{i=0,\dots,d} (g_i(t)Z_{s_i}), \quad t \in [0, 1].$$

Denote by $\|\cdot\|_{D_{i-1,i}}$ the D -norm pertaining to the bivariate rv $(\eta_{s_{i-1}}, \eta_{s_i})$, $i = 1, \dots, d$. Define the functions g_0, \dots, g_d by

$$\begin{aligned} g_0(t) &:= \begin{cases} \frac{s_1 - t}{\|(s_1 - t, t)\|_{D_{0,1}}}, & t \in [0, s_1], \\ 0, & \text{else,} \end{cases} \\ g_i(t) &:= \begin{cases} \frac{t - s_{i-1}}{\|(s_i - t, t - s_{i-1})\|_{D_{i-1,i}}}, & t \in [s_{i-1}, s_i], \\ \frac{s_{i+1} - t}{\|(s_{i+1} - t, t - s_i)\|_{D_{i,i+1}}}, & t \in [s_i, s_{i+1}], \quad i = 1, \dots, d-1, \\ 0, & \text{else,} \end{cases} \\ g_d(t) &:= \begin{cases} \frac{t - s_{d-1}}{\|(s_d - t, t - s_{d-1})\|_{D_{d-1,d}}}, & t \in [s_{d-1}, s_d], \\ 0, & \text{else.} \end{cases} \end{aligned}$$

It is obvious that $g_0, \dots, g_d \in \bar{C}^+[0, 1]$. Moreover, we have for $t \in [s_{i-1}, s_i]$, $i = 1, \dots, d$,

$$\|(g_0(t), \dots, g_d(t))\|_{D_{0,\dots,d}} = \|(g_{i-1}(t), g_i(t))\|_{D_{i-1,i}} = 1.$$

Hence, the functions g_0, \dots, g_d are suitable for the generalized max-linear model (2). In addition, they have the following property:

Lemma 3.1. *The functions g_0, \dots, g_d defined above satisfy*

$$\|g_i\|_{\infty} = g_i(s_i) = 1, \quad i = 0, \dots, d.$$

Proof. From the fact that a D -norm is monotone and standardized we obtain for $i = 1, \dots, d-1$ and $t \in [s_{i-1}, s_i]$

$$g_i(t) = \frac{t - s_{i-1}}{\|(s_i - t, t - s_{i-1})\|_{D_{i-1,i}}} = \frac{1}{\left\| \left(\frac{s_i - t}{t - s_{i-1}}, 1 \right) \right\|_{D_{i-1,i}}} \leq \frac{1}{\|(0, 1)\|_{D_{i-1,i}}} = 1,$$

and for $t \in [s_i, s_{i+1}]$

$$g_i(t) = \frac{s_{i+1} - t}{\|(s_{i+1} - t, t - s_i)\|_{D_{i,i+1}}} = \frac{1}{\left\| \left(1, \frac{t - s_i}{s_{i+1} - t} \right) \right\|_{D_{i,i+1}}} \leq \frac{1}{\|(1, 0)\|_{D_{i,i+1}}} = 1.$$

Analogously, we have $g_0 \leq 1$ and $g_d \leq 1$. The assertion now follows since $g_i(s_i) = 1$, $i = 0, \dots, d$. \square

The SMSP $\hat{\eta} = (\hat{\eta}_t)_{t \in [0,1]}$ that is generated by the generalized max-linear model with these particular functions g_0, \dots, g_d is given by

$$(4) \quad \begin{aligned} \hat{\eta}_t &= \max \left(\frac{\eta_{s_{i-1}}}{g_{i-1}(t)}, \frac{\eta_{s_i}}{g_i(t)} \right) \\ &= \|(s_i - t, t - s_{i-1})\|_{D_{i-1,i}} \max \left(\frac{\eta_{s_{i-1}}}{s_i - t}, \frac{\eta_{s_i}}{t - s_{i-1}} \right), \quad t \in [s_{i-1}, s_i], \quad i = 1, \dots, d. \end{aligned}$$

Note that $\eta_{s_i} < 0$ almost sure, $i = 0, \dots, d$. This implies that the maximum taken over $d+1$ points in (2) goes down to a maximum taken over only two points in (4) since all except two of the g_i vanish in $t \in [s_{i-1}, s_i]$, $i = 1, \dots, d$.

The above process obviously interpolates the rv $(\eta_{s_0}, \dots, \eta_{s_d})$. In summary, we have proven the following result.

Corollary 3.2. *Let $\eta = (\eta_t)_{t \in [0,1]}$ be an SMSP with generator $\mathbf{Z} = (Z_t)_{t \in [0,1]}$, and let $0 := s_0 < s_1 < \dots < s_{d-1} < s_d := 1$ be a grid of points in the interval $[0, 1]$. The process $\hat{\eta} = (\hat{\eta}_t)_{t \in [0,1]}$ defined in (4) is an SMSP with generator process $\hat{\mathbf{Z}} = (\hat{Z}_t)_{t \in [0,1]}$, where*

$$(5) \quad \hat{Z}_t = \max \left(g_{i-1}(t)Z_{s_{i-1}}, g_i(t)Z_{s_i} \right), \quad t \in [s_{i-1}, s_i], \quad i = 1, \dots, d.$$

The processes $\hat{\eta}$ and $\hat{\mathbf{Z}}$ interpolate the rv $(\eta_{s_0}, \dots, \eta_{s_d})$ and $(Z_{s_0}, \dots, Z_{s_d})$, respectively.

We call $\hat{\eta}$ the discretized version of η and $\hat{\mathbf{Z}}$ the discretized version of \mathbf{Z} , both with grid $\{s_0, \dots, s_d\}$. We show now that the preceding approach is suitable to approximate the underlying SMSP solely by multivariate observations; that is, the discretized version of the underlying SMSP converges to this very process in a strong sense. We need the following two lemmata which provide some technical insight in the structure of the chosen max-linear model.

Lemma 3.3. *The SMSP defined in (4) fulfills for $i = 1, \dots, d$*

$$\sup_{t \in [s_{i-1}, s_i]} \hat{\eta}_t = \max(\eta_{s_{i-1}}, \eta_{s_i}),$$

and

$$\inf_{t \in [s_{i-1}, s_i]} \hat{\eta}_t = -\|(\eta_{s_{i-1}}, \eta_{s_i})\|_{D_{s_{i-1}, s_i}}.$$

This minimum is attained for $t = (s_{i-1}\eta_{s_{i-1}} + s_i\eta_{s_i})/(\eta_{s_{i-1}} + \eta_{s_i})$.

Proof. We know from Lemma 3.1 that $g_{i-1}(t), g_i(t) \leq 1$ for an arbitrary $i = 1, \dots, d$ and $t \in [s_{i-1}, s_i]$. Hence,

$$\hat{\eta}_t = \max \left(\frac{\eta_{s_{i-1}}}{g_{i-1}(t)}, \frac{\eta_{s_i}}{g_i(t)} \right) \leq \max(\eta_{s_{i-1}}, \eta_{s_i})$$

for $i = 1, \dots, d$ and $t \in [s_{i-1}, s_i]$, which yields the first part of the assertion. Recall that $\eta_{s_i} < 0$ with probability one, $i = 0, \dots, d$.

Moreover, we have for $t \in (s_{i-1}, s_i)$

$$\frac{\eta_{s_{i-1}}}{s_i - t} \leq \frac{\eta_{s_i}}{t - s_{i-1}} \iff \frac{s_i - t}{t - s_{i-1}} \leq \frac{\eta_{s_{i-1}}}{\eta_{s_i}} \iff t \geq \frac{s_{i-1}\eta_{s_{i-1}} + s_i\eta_{s_i}}{\eta_{s_{i-1}} + \eta_{s_i}},$$

where equality in one of these expressions occurs iff it does in the other two. In this case of equality we have

$$\hat{\eta}_t = \|(s_i - t, t - s_{i-1})\|_{D_{i-1,i}} \cdot \frac{\eta_{s_i}}{t - s_{i-1}} = -\|(\eta_{s_{i-1}}, \eta_{s_i})\|_{D_{i-1,i}}.$$

On the other hand, the monotonicity of a D -norm implies for every $t \in (s_{i-1}, s_i)$ with $t \geq (s_{i-1}\eta_{s_{i-1}} + s_i\eta_{s_i})/(\eta_{s_{i-1}} + \eta_{s_i})$

$$\begin{aligned} \hat{\eta}_t &= \|(s_i - t, t - s_{i-1})\|_{D_{i-1,i}} \cdot \frac{\eta_{s_i}}{t - s_{i-1}} \\ &= \left\| \left(\frac{s_i - t}{t - s_{i-1}}, 1 \right) \right\|_{D_{i-1,i}} \cdot \eta_{s_i} \\ &\geq \left\| \left(\frac{\eta_{s_{i-1}}}{\eta_{s_i}}, 1 \right) \right\|_{D_{i-1,i}} \cdot \eta_{s_i} \\ &= -\|(\eta_{s_{i-1}}, \eta_{s_i})\|_{D_{i-1,i}}. \end{aligned}$$

Recall again that $\eta_{s_i} < 0$ almost sure. The case $t \leq (s_{i-1}\eta_{s_{i-1}} + s_i\eta_{s_i})/(\eta_{s_{i-1}} + \eta_{s_i})$ works analogously. \square

As an immediate consequence of the preceding result we obtain that for $x \leq 0$

$$\hat{\eta} \leq x \iff \max(\eta_{s_0}, \dots, \eta_{s_d}) \leq x$$

and

$$\hat{\eta} > x \iff \max_{1 \leq i \leq d} \|(\eta_{s_{i-1}}, \eta_{s_i})\|_{D_{i-1,i}} < -x.$$

The next lemma is on the structure of the underlying generator processes.

Lemma 3.4. *The generator process defined in (5) fulfills for $i = 1, \dots, d$*

$$\sup_{t \in [s_{i-1}, s_i]} \hat{Z}_t = \max(Z_{s_{i-1}}, Z_{s_i}).$$

In particular, the extremal coefficient $E(\|\hat{Z}\|_\infty)$ of the SMSP $\hat{\eta}$ coincides with the extremal coefficient $E(\max_{i=0, \dots, d} Z_{s_i})$ of the rv $(\eta_{s_0}, \dots, \eta_{s_d})$. Moreover, for $i = 1, \dots, d$,

$$\inf_{t \in [s_{i-1}, s_i]} \hat{Z}_t = \begin{cases} \left(\|(1/Z_{s_{i-1}}, 1/Z_{s_i})\|_{D_{i-1,i}} \right)^{-1} & \text{if } Z_{s_{i-1}}, Z_{s_i} > 0, \\ 0 & \text{else.} \end{cases}$$

In the first case, the minimum is attained for $t = (s_{i-1}Z_{s_i} + s_iZ_{s_{i-1}})/(Z_{s_{i-1}} + Z_{s_i})$.

Proof. We know from Lemma 3.1 that $g_{i-1}(t), g_i(t) \leq 1$ for an arbitrary $i = 1, \dots, d$ and $t \in [s_{i-1}, s_i]$. Hence,

$$\hat{Z}_t = \max \left(g_{i-1}(t)Z_{s_{i-1}}, g_i(t)Z_{s_i} \right) \leq \max(Z_{s_{i-1}}, Z_{s_i})$$

for $i = 1, \dots, d$ and $t \in [s_{i-1}, s_i]$, which yields the first part of the assertion.

Moreover, we have for $t \in (s_{i-1}, s_i)$ in case of $Z_{s_{i-1}}, Z_{s_i} > 0$

$$(s_i - t)Z_{s_{i-1}} \leq (t - s_{i-1})Z_{s_i} \iff \frac{s_i - t}{t - s_{i-1}} \leq \frac{Z_{s_i}}{Z_{s_{i-1}}} \iff t \geq \frac{s_{i-1}Z_{s_i} + s_iZ_{s_{i-1}}}{Z_{s_{i-1}} + Z_{s_i}},$$

where equality in one of these expressions occurs iff it does in the other two. In this case of equality we have

$$\hat{Z}_t = \frac{(t - s_{i-1})Z_{s_i}}{\|(s_i - t, t - s_{i-1})\|_{D_{i-1,i}}} = \frac{1}{\|(1/Z_{s_{i-1}}, 1/Z_{s_i})\|_{D_{i-1,i}}}.$$

On the other hand, the monotonicity of a D -norm implies for every $t \in (s_{i-1}, s_i)$ with $t \geq (s_{i-1}Z_{s_i} + s_iZ_{s_{i-1}})/(Z_{s_{i-1}} + Z_{s_i})$

$$\begin{aligned} \hat{Z}_t &= \frac{(t - s_{i-1})Z_{s_i}}{\|(s_i - t, t - s_{i-1})\|_{D_{i-1,i}}} \\ &= \left(\left\| \left(\frac{s_i - t}{t - s_{i-1}}, 1 \right) \right\|_{D_{i-1,i}} \right)^{-1} Z_{s_i} \\ &\geq \left(\left\| \left(\frac{Z_{s_i}}{Z_{s_{i-1}}}, 1 \right) \right\|_{D_{i-1,i}} \right)^{-1} Z_{s_i} \\ &= \frac{1}{\|(1/Z_{s_{i-1}}, 1/Z_{s_i})\|_{D_{i-1,i}}}. \end{aligned}$$

The case $t \leq (s_{i-1}Z_{s_i} + s_iZ_{s_{i-1}})/(Z_{s_{i-1}} + Z_{s_i})$ is shown analogously. \square

So far we have only considered a fixed discretized version of an SMSP. The next step is to examine a sequence of discretized versions with certain grids whose fineness converges to zero. It turns out that such a sequence converges to the initial SMSP in the function space $C[0, 1]$ equipped with the sup-norm. Thus, our method is suitable to reconstruct the initial process.

Let

$$\mathcal{G}_d := \{s_0^{(d)}, s_1^{(d)}, \dots, s_d^{(d)}\}, \quad 0 =: s_0^{(d)} < s_1^{(d)} < \dots < s_d^{(d)} := 1, \quad d \in \mathbb{N},$$

be a sequence of grids in $[0, 1]$ with fineness

$$\kappa_d := \max_{i=1, \dots, d} \left(s_i^{(d)} - s_{i-1}^{(d)} \right) \rightarrow_{d \rightarrow \infty} 0.$$

Let $\hat{\boldsymbol{\eta}}^{(d)} = (\hat{\eta}_t^{(d)})_{t \in [0,1]}$ be the discretized version of an SMSP $\boldsymbol{\eta} = (\eta_t)_{t \in [0,1]}$ with grid \mathcal{G}_d . Denote by $\hat{\mathbf{Z}}^{(d)} = (\hat{Z}_t^{(d)})_{t \in [0,1]}$ and $\mathbf{Z} = (Z_t)_{t \in [0,1]}$ the generator processes pertaining to $\hat{\boldsymbol{\eta}}^{(d)}$ and $\boldsymbol{\eta}$, respectively.

Theorem 3.5. *The processes $\hat{\boldsymbol{\eta}}^{(d)}$ and $\hat{\mathbf{Z}}^{(d)}$, $d \in \mathbb{N}$, converge uniformly to $\boldsymbol{\eta}$ and \mathbf{Z} , respectively, i. e. $\|\hat{\boldsymbol{\eta}}^{(d)} - \boldsymbol{\eta}\|_\infty \rightarrow_{d \rightarrow \infty} 0$ and $\|\hat{\mathbf{Z}}^{(d)} - \mathbf{Z}\|_\infty \rightarrow_{d \rightarrow \infty} 0$.*

Proof. Denote by $[t]_d$, $d \in \mathbb{N}$, the left neighbor of $t \in [0,1]$ among \mathcal{G}_d , and by $\langle t \rangle_d$, $d \in \mathbb{N}$, the right neighbor of $t \in [0,1]$ among \mathcal{G}_d . Choose a sequence $s^{(d)} \in [0,1]$, $d \in \mathbb{N}$, with $s^{(d)} \rightarrow_{d \in \mathbb{N}} s \in [0,1]$. Then obviously $[s^{(d)}]_d \rightarrow_{d \rightarrow \infty} s$ and $\langle s^{(d)} \rangle_d \rightarrow_{d \rightarrow \infty} s$. Hence we obtain by Lemma 3.3, and the continuity of the process $\boldsymbol{\eta}$

$$\hat{\eta}_{s^{(d)}}^{(d)} \leq \max_{s \in [[s^{(d)}]_d, \langle s^{(d)} \rangle_d]} \hat{\eta}_s^{(d)} = \max(\eta_{[s^{(d)}]_d}, \eta_{\langle s^{(d)} \rangle_d}) \rightarrow_{d \rightarrow \infty} \eta_s,$$

as well as

$$\hat{\eta}_{s^{(d)}}^{(d)} \geq \min_{s \in [[s^{(d)}]_d, \langle s^{(d)} \rangle_d]} \hat{\eta}_s^{(d)} = -\|(\eta_{[s^{(d)}]_d}, \eta_{\langle s^{(d)} \rangle_d})\|_{D_{[s^{(d)}]_d, \langle s^{(d)} \rangle_d}} \rightarrow_{d \rightarrow \infty} \eta_s,$$

which yields the first part of the assertion.

Now we show that $\hat{\mathbf{Z}}^{(d)} \rightarrow_{d \rightarrow \infty} \mathbf{Z}$ in $(C[0,1], \|\cdot\|_\infty)$. If $Z_s \neq 0$, the continuity of \mathbf{Z} implies $Z_{[s^{(d)}]_d} \neq 0 \neq Z_{\langle s^{(d)} \rangle_d}$ for sufficiently large values of d . Repeating the above arguments, the assertion now follows by Lemma 3.4. If $Z_s = 0$, the continuity of \mathbf{Z} implies

$$\hat{Z}_{s^{(d)}}^{(d)} \leq 2 \max(Z_{[s^{(d)}]_d}, Z_{\langle s^{(d)} \rangle_d}) \rightarrow_{d \rightarrow \infty} 2Z_s = 0,$$

which completes the proof. Check that $\|(\langle s^{(d)} \rangle_d - t, t - [s^{(d)}]_d)\|_D \geq 1/2$ since every D -norm is monotone and standardized. \square

The preceding theorem is the main reason why we consider the discretized version $\hat{\boldsymbol{\eta}}$ of an SMSP $\boldsymbol{\eta}$ a reasonable predictor of this process. The predictions $\hat{\eta}_t$ of the points η_t , $t \in [0,1]$, only depend on the multivariate observations $(\eta_{s_0}, \dots, \eta_{s_d})$. More precisely, the only thing we need to know to make these predictions is the set of the adjacent bivariate marginal distributions of $(\eta_{s_0}, \dots, \eta_{s_d})$, that is, the bivariate D -norms $\|\cdot\|_{D_{i-1,i}}$, $i = 1, \dots, d$. Since it is assumed that the values η_{s_i} , $i = 0, \dots, d$ are known, the prediction $\hat{\eta}_t$ usefully coincides with the actual value η_t for $t \in \{s_0, \dots, s_d\}$.

3.2. The Mean Squared Error of the Discretized Version. Considering $\hat{\eta}_t$ a predictor of η_t we can ask for further properties such as the mean squared error, which is our next aim. For that purpose, we formulate the following lemma. It applies to bivariate standard max-stable rv in general.

Lemma 3.6. *Let (X, Y) be a bivariate standard max-stable rv, i. e. there exists some D -norm $\|\cdot\|_D$ such that $P(X \leq x, Y \leq y) = \exp(-\|(x, y)\|_D)$, $x, y \leq 0$. Then*

$$E(XY) = \int_0^\infty \frac{1}{\|(1, t)\|_D^2} dt.$$

In particular, the covariance and the correlation coefficient ϱ of X and Y are given by

$$\text{Cov}(X, Y) = \int_0^\infty \frac{1}{\|(1, t)\|_D^2} dt - 1 = \varrho(X, Y).$$

Proof. Recall that the expected value of a negative exponentially distributed random variable ξ with parameter $\lambda > 0$ is given by

$$E(\xi) = \int_{-\infty}^0 x \lambda \exp(\lambda x) dx = -1/\lambda.$$

Hence, elementary calculations show

$$\begin{aligned} E(XY) &= \int_{-\infty}^0 \int_{-\infty}^0 P(X \leq x, Y \leq y) dx dy \\ &= \int_{-\infty}^0 \int_{-\infty}^0 \exp(-\|(x, y)\|_D) dx dy \\ &= \int_{-\infty}^0 \int_{-\infty}^0 \exp(x \|(1, y/x)\|_D) dx dy \\ &= - \int_0^\infty \int_{-\infty}^0 x \exp(x \|(1, u)\|_D) dx du \\ &= - \int_0^\infty \frac{1}{\|(1, u)\|_D} \int_{-\infty}^0 x \|(1, u)\|_D \exp(x \|(1, u)\|_D) dx du \\ &= \int_0^\infty \frac{1}{\|(1, u)\|_D^2} du. \end{aligned}$$

The remaining assertions follow from the fact that $E(X) = E(Y) = -1$ and $\text{Var}(X) = \text{Var}(Y) = 1$. \square

EXAMPLE 3.7. In accordance to the characterization of the independence and complete dependence case in terms of D -norms, we obtain in case of $\|\cdot\|_D = \|\cdot\|_1$

$$\text{Cov}(X, Y) = \int_0^\infty \frac{1}{(u+1)^2} du - 1 = 0$$

and in case of $\|\cdot\|_D = \|\cdot\|_\infty$

$$\text{Cov}(X, Y) = \int_0^\infty \frac{1}{(\max(u, 1))^2} du - 1 = 1.$$

In particular, we have $\text{Cov}(X, Y) = \varrho(X, Y) \in [0, 1]$ for every bivariate standard max-stable rv (X, Y) since the maximum norm is the least D -norm and the sum

norm is the largest D -norm. In addition to this, we obtain for $\|\cdot\|_D = \|\cdot\|_2$

$$\text{Cov}(X, Y) = \int_0^\infty \frac{1}{(u^2 + 1)} du - 1 = \left[\arctan(u) \right]_0^\infty - 1 = \pi/2 - 1.$$

In order to calculate the mean squared error of the predictor $\hat{\eta}_t$, we have to determine the mixed moment $E(\eta_t \hat{\eta}_t)$. For this purpose, we want to apply the previous lemma which is why we need to ensure that the vector $(\eta_t, \hat{\eta}_t)$ is standard max-stable itself. This is the content of the following Lemma.

Lemma 3.8. *Let $\boldsymbol{\eta} = (\eta_t)_{t \in [0,1]}$ be an SMSP and denote by $\hat{\boldsymbol{\eta}} = (\hat{\eta}_t)_{t \in [0,1]}$ its discretized version with grid $\{s_0, \dots, s_d\}$. Then the bivariate rv $(\eta_t, \hat{\eta}_t)$ is standard max-stable for every $t \in [0, 1]$ with the pertaining D -norm*

$$\|(x, y)\|_{D_t} := \left\| \left(x, g_{i-1}(t)y, g_i(t)y \right) \right\|_{D_{t, i-1, i}}, \quad t \in [s_{i-1}, s_i], \quad i = 1, \dots, d,$$

where $\|\cdot\|_{D_{t, i-1, i}}$ is the D -norm pertaining to $(\eta_t, \eta_{s_{i-1}}, \eta_{s_i})$.

Proof. We have for every $t \in [s_{i-1}, s_i]$, $x, y \leq 0$ and $i = 1, \dots, d$

$$\begin{aligned} P(\eta_t \leq x, \hat{\eta}_t \leq y) &= P(\eta_t \leq x, \eta_{s_{i-1}} \leq g_{i-1}(t)y, \eta_{s_i} \leq g_i(t)y) \\ &= \exp \left(-E \left(\max \left(|x| Z_t, g_{i-1}(t) |y| Z_{s_{i-1}}, g_i(t) |y| Z_{s_i} \right) \right) \right) \\ &= \exp \left(-E \left(\max \left(|x| Z_t, |y| \max (g_{i-1}(t) Z_{s_{i-1}}, g_i(t) Z_{s_i}) \right) \right) \right). \end{aligned}$$

The vector

$$\left(Z_t, \max (g_{i-1}(t) Z_{s_{i-1}}, g_i(t) Z_{s_i}) \right)$$

defines a generator for every $t \in [s_{i-1}, s_i]$, $i = 1, \dots, d$ as for all such t

$$E \left(\max (g_{i-1}(t) Z_{s_{i-1}}, g_i(t) Z_{s_i}) \right) = \|(g_{i-1}(t), g_i(t))\|_{D_{i-1, i}} = 1.$$

□

Let us recall the sequence of processes we have discussed in Theorem 3.5. Suppose $\boldsymbol{\eta}$ is an SMSP and choose a sequence of grids \mathcal{G}_d of the interval $[0, 1]$ with fineness $\kappa_d \rightarrow_{d \rightarrow \infty} 0$. Denote by $\hat{\boldsymbol{\eta}}^{(d)}$, $d \in \mathbb{N}$, the sequence of discretized versions of $\boldsymbol{\eta}$ with grid \mathcal{G}_d . Denote further by $\|\cdot\|_{D_t^{(d)}}$ the D -norm pertaining to $(\eta_t, \hat{\eta}_t^{(d)})$, $t \in [0, 1]$, $d \in \mathbb{N}$.

Theorem 3.9. *Let $\boldsymbol{\eta}$ and $\hat{\boldsymbol{\eta}}^{(d)}$, $d \in \mathbb{N}$, be as above. The mean squared error of $\hat{\eta}_t^{(d)}$ is given by*

$$\text{MSE} \left(\hat{\eta}_t^{(d)} \right) := E \left(\left(\eta_t - \hat{\eta}_t^{(d)} \right)^2 \right) = 2 \left(2 - \int_0^\infty \frac{1}{\|(1, u)\|_{D_t^{(d)}}^2} du \right) \rightarrow_{d \rightarrow \infty} 0.$$

Proof. The second moment of a standard negative exponentially distributed random variable is two and, therefore, we obtain by Lemma 3.6 and Lemma 3.8

$$\begin{aligned} \text{MSE} \left(\eta_t^{(d)} \right) &= E \left(\left(\eta_t - \hat{\eta}_t^{(d)} \right)^2 \right) \\ &= E \left(\eta_t^2 \right) - 2E \left(\eta_t \hat{\eta}_t^{(d)} \right) + E \left(\left(\hat{\eta}_t^{(d)} \right)^2 \right) \\ &= 4 - 2 \int_0^\infty \frac{1}{\|(1, u)\|_{D_t^{(d)}}^2} du. \end{aligned}$$

Next, we show $\|\cdot\|_{D_t^{(d)}} \rightarrow_{d \rightarrow \infty} \|\cdot\|_\infty$ for all $t \in [0, 1]$. Denote by \mathbf{Z} and $\hat{\mathbf{Z}}^{(d)}$, $d \in \mathbb{N}$, the generator processes of $\boldsymbol{\eta}$ and $\hat{\boldsymbol{\eta}}^{(d)}$, $d \in \mathbb{N}$. Define

$$m := E \left(\sup_{t \in [0, 1]} Z_t \right) < \infty \quad \text{and} \quad \tilde{Z} := \frac{\sup_{t \in [0, 1]} Z_t}{m}.$$

Then $E(\tilde{Z}) = 1$ and, thus, (Z_t, \tilde{Z}) defines a generator for all $t \in [0, 1]$. Denote by $\|\cdot\|_{\tilde{D}}$ the D -norm pertaining to this generator. Lemma 3.1 and Corollary 3.2 imply $\hat{\mathbf{Z}}^{(d)} \leq \mathbf{Z}$ for all $d \in \mathbb{N}$. Therefore, we have for arbitrary $x, y \in \mathbb{R}$, $d \in \mathbb{N}$ and $t \in [0, 1]$

$$\max \left(|x| Z_t, |y| \hat{Z}_t^{(d)} \right) \leq \max \left(|x| Z_t, |my| \tilde{Z}_t \right),$$

where

$$E \left(\max \left(|x| Z_t, |my| \tilde{Z}_t \right) \right) = \|(x, my)\|_{\tilde{D}} < \infty.$$

Hence, the dominated convergence theorem, together with the fact that $\hat{Z}_t^{(d)} \rightarrow_{d \rightarrow \infty} Z_t$ for all $t \in [0, 1]$ by Theorem 3.5, implies

$$\|(x, y)\|_{D_t^{(d)}} = E \left(\max \left(|x| Z_t, |y| \hat{Z}_t^{(d)} \right) \right) \rightarrow_{d \rightarrow \infty} E \left(\max \left(|x| Z_t, |y| Z_t \right) \right) = \|(x, y)\|_\infty$$

for all $x, y \in \mathbb{R}$.

In Example 3.7, we have already calculated $\int_0^\infty \|(1, u)\|_\infty^{-2} du = 2$. Since $\|\cdot\|_\infty$ is the least D -norm, we have for all $d \in \mathbb{N}$ and $t \in [0, 1]$

$$\frac{1}{\|(1, u)\|_{D_t^{(d)}}^2} \leq \frac{1}{\|(1, u)\|_\infty^2},$$

and therefore by the dominated convergence theorem again

$$\int_0^\infty \frac{1}{\|(1, u)\|_{D_t^{(d)}}^2} du \rightarrow_{d \rightarrow \infty} \int_0^\infty \frac{1}{\|(1, u)\|_\infty^2} du = 2,$$

which completes the proof. \square

4. RECONSTRUCTION OF SGPP

The preceding technique of discretizing and reconstructing a given SMSP can also be applied to the case of SGPP by simply replacing the standard max-stable rv in the model (2) by a standard generalized Pareto distributed rv. Again, the generalized max-linear model results in an SGPP. Once this statement is proven, most of the results of the previous sections carry over in a very straightforward way.

Recall that a stochastic process \mathbf{V} in $\bar{C}^-[0, 1]$ is an SGPP, if there exists a D -norm $\|\cdot\|_D$ on $E[0, 1]$ and some $c_0 > 0$, such that $P(\mathbf{V} \leq f) = 1 - \|f\|_D$ for all $f \in \bar{E}^-[0, 1]$ with $\|f\|_\infty \leq c_0$.

4.1. Uniform Convergence of the Discretized Versions. Let (Y_0, Y_1, \dots, Y_d) follow a standard generalized Pareto distribution (GPD), i.e. there is a D -norm $\|\cdot\|_{D_0, \dots, d}$ on \mathbb{R}^{d+1} generated by (Z_0, \dots, Z_d) , and a vector $\mathbf{y}^{(0)} = (y_0^{(0)}, \dots, y_d^{(0)}) < \mathbf{0}$, such that $P(Y_0 \leq y_0, \dots, Y_d \leq y_d) = 1 - \|\mathbf{y}\|_{D_0, \dots, d}$ for all $\mathbf{y} = (y_0, \dots, y_d)$ with $\mathbf{y}^{(0)} \leq \mathbf{y} \leq \mathbf{0}$. Note that this implies that each univariate marginal distribution of (Y_0, \dots, Y_d) coincides in the upper tail with the uniform distribution on $[-1, 0]$. For a detailed examination of GPD rv, see e.g. Falk et al. (2010).

We apply the generalized max-linear model to the GPD rv and obtain a stochastic process $\mathbf{V} = (V_t)_{t \in [0, 1]}$,

$$(6) \quad V_t := \max_{i=0, \dots, d} \frac{Y_i}{g_i(t)},$$

where $g_0, \dots, g_d \in \bar{C}^+[0, 1]$ are functions satisfying condition (1). Again, this process defines an SGPP as the next lemma shows.

Lemma 4.1. *The stochastic process $\mathbf{V} = (V_t)_{t \in [0, 1]}$ in (6) defines an SGPP with generator process $\hat{\mathbf{Z}} = (\hat{Z}_t)_{t \in [0, 1]}$,*

$$\hat{Z}_t = \max_{i=0, \dots, d} (g_i(t) Z_i), \quad t \in [0, 1].$$

Proof. We have already shown in Lemma 2.1 that $\hat{\mathbf{Z}}$ defines a generator process. Put $c_0 := -\max_{j=0, \dots, d} (y_j^{(0)} / \|g_j\|_\infty)$. Then we have for all $f \in \bar{E}^-[0, 1]$

$$\|f\|_\infty \leq c_0 \iff \inf_{t \in [0, 1]} f(t) \geq \max_{j=0, \dots, d} (y_j^{(0)} / \|g_j\|_\infty)$$

and therefore for $i = 0, \dots, d$

$$\begin{aligned} \inf_{t \in [0, 1]} (g_i(t) f(t)) &\geq \inf_{t \in [0, 1]} \left(g_i(t) \max_{j=0, \dots, d} \left(\frac{y_j^{(0)}}{\|g_j\|_\infty} \right) \right) \\ &= \max_{j=0, \dots, d} \left(\frac{\inf_{t \in [0, 1]} g_i(t)}{\sup_{t \in [0, 1]} g_j(t)} \cdot y_j^{(0)} \right) \end{aligned}$$

$$\begin{aligned}
&\geq \frac{\inf_{t \in [0,1]} g_i(t)}{\sup_{t \in [0,1]} g_i(t)} \cdot y_i^{(0)} \\
&\geq y_i^{(0)}.
\end{aligned}$$

Hence, we have for all f close enough to zero

$$\begin{aligned}
P(\mathbf{V} \leq f) &= P(Y_i \leq g_i(t)f(t), \ i = 0, \dots, d, \ t \in [0, 1]) \\
&= P\left(Y_i \leq \inf_{t \in [0,1]} (g_i(t)f(t)), \ i = 0, \dots, d\right) \\
&= 1 - \left\| \left(\sup_{t \in [0,1]} (g_0 |f(t)|), \dots, \sup_{t \in [0,1]} (g_d |f(t)|) \right) \right\|_{D_0, \dots, d} \\
&= 1 - E \left(\max_{i=0, \dots, d} \left(\sup_{t \in [0,1]} (g_i(t) |f(t)|) Z_i \right) \right) \\
&= 1 - E \left(\sup_{t \in [0,1]} \left(|f(t)| \max_{i=0, \dots, d} (g_i(t) Z_i) \right) \right) \\
&= 1 - E \left(\sup_{t \in [0,1]} (|f(t)| Z'_t) \right),
\end{aligned}$$

which completes the proof. \square

Just like in the SMSP case, we can use this model to discretize and reconstruct a given SGPP. Let $\mathbf{V} = (V_t)_{t \in [0,1]}$ be an SGPP with generator process $\mathbf{Z} = (Z_t)_{t \in [0,1]}$ and D -norm $\|\cdot\|_D$. Choose a grid $0 =: s_0 < s_1 < \dots < s_{d-1} < s_d := 1$ of points within $[0, 1]$. Then $(V_{s_0}, \dots, V_{s_d})$ is a standard GPD rv in \mathbb{R}^{d+1} with pertaining D -norm $\|\cdot\|_{D_0, \dots, d}$ generated by $(Z_{s_0}, \dots, Z_{s_d})$.

Now choose deterministic functions $g_0, \dots, g_d \in \bar{C}^+[0, 1]$ with the property (1) and put for $t \in [0, 1]$

$$(7) \quad \hat{V}_t := \max_{i=0, \dots, d} \frac{V_{s_i}}{g_i(t)}.$$

As we have shown in Lemma 4.1, the stochastic process $\hat{\mathbf{V}} = (\hat{V}_t)_{t \in [0,1]}$ in (7) defines an SGPP with generator process $\hat{\mathbf{Z}} = (\hat{Z}_t)_{t \in [0,1]}$,

$$\hat{Z}_t = \max_{i=0, \dots, d} (g_i(t) Z_{s_i}), \quad t \in [0, 1].$$

By choosing the exact same functions g_0, \dots, g_d as in the special case of the generalized max-linear model in Section 3, we obtain the process

$$\begin{aligned}
(7') \quad \hat{V}_t &= \max \left(\frac{V_{s_{i-1}}}{g_{i-1}(t)}, \frac{V_{s_i}}{g_i(t)} \right) \\
&= \|(s_i - t, t - s_{i-1})\|_{D_{i-1,i}} \max \left(\frac{V_{s_{i-1}}}{s_i - t}, \frac{V_{s_i}}{t - s_{i-1}} \right), \quad t \in [s_{i-1}, s_i], \ i = 1, \dots, d.
\end{aligned}$$

In order to show that this process defines an SGPP as well, we only have to verify that the functions g_0, \dots, g_d realize in $\bar{C}^+[0, 1]$ and satisfy condition (1), which we have already done in Section 3. Thus, the following result is proven.

Corollary 4.2. *Let $\mathbf{V} = (\eta_t)_{t \in [0, 1]}$ be an SGPP with generator $\mathbf{Z} = (Z_t)_{t \in [0, 1]}$, and $0 := s_0 < s_1 < \dots < s_{d-1} < s_d := 1$ be a grid in the interval $[0, 1]$. The process $\hat{\mathbf{V}} = (\hat{V}_t)_{t \in [0, 1]}$ defined in (7') is an SGPP with generator process $\hat{\mathbf{Z}} = (\hat{Z}_t)_{t \in [0, 1]}$, where*

$$\hat{Z}_t = \max(g_{i-1}(t)Z_{s_{i-1}}, g_i(t)Z_{s_i}), \quad t \in [s_{i-1}, s_i], \quad i = 1, \dots, d.$$

The processes $\hat{\mathbf{V}}$ and $\hat{\mathbf{Z}}$ interpolate the rv $(V_{s_0}, \dots, V_{s_d})$ and $(Z_{s_0}, \dots, Z_{s_d})$, respectively.

In complete accordance to the SMSP case we call $\hat{\mathbf{V}}$ the discretized version of \mathbf{V} with grid $\{s_0, \dots, s_d\}$.

REMARK 4.3. Let the original SGPP \mathbf{V} satisfy $P(\mathbf{V} \leq f) = 1 - \|f\|_D$ for all $f \in \bar{E}^-[0, 1]$ with $\|f\|_\infty \leq c_0$, with some D -norm $\|\cdot\|_D$ and some $c_0 > 0$. It is clear that this implies

$$P(V_{s_0} \leq y_0, \dots, V_{s_d} \leq y_d) = 1 - \|(y_0, \dots, y_d)\|_{D_{0, \dots, d}}$$

for all $\mathbf{y} := (y_0, \dots, y_d)$ with $-c_0 \leq y_i \leq 0$, $i = 0, \dots, d$. Now denote by $\|\cdot\|_{\hat{D}}$ the D -norm pertaining to the discretized version $\hat{\mathbf{V}}$ of \mathbf{V} with grid $\{s_0, \dots, s_d\}$. Just like in the proof of Lemma 4.1 we obtain $P(\hat{\mathbf{V}} \leq f) = 1 - \|f\|_{\hat{D}}$ for all $f \in \bar{E}^-[0, 1]$ with $\|f\|_\infty \leq \hat{c}_0$, where

$$\hat{c}_0 = - \max_{i=0, \dots, d} \left(\frac{-c_0}{\|g_i\|_\infty} \right) = c_0,$$

since in this special case $\|g_i\|_\infty = 1$ holds for all $i = 0, \dots, d$ according to Lemma 3.1. Thus, the upper tail where we know the distribution of the discretized version $\hat{\mathbf{V}}$ coincides with that of the initial SGPP \mathbf{V} .

It is obvious that the pathwise structure of the discretized version of an SMSP we have established in Lemma 3.3 now carries over to the SGPP case since the assertion in this lemma solely follows from the structure of g_0, \dots, g_d and the fact that the initial process is nonpositive with probability one.

Lemma 4.4. *The SGPP defined in (7') fulfills for $i = 1, \dots, d$*

$$\sup_{t \in [s_{i-1}, s_i]} \hat{V}_t = \max(V_{s_{i-1}}, V_{s_i}),$$

and

$$\inf_{t \in [s_{i-1}, s_i]} \hat{V}_t = -\|(V_{s_{i-1}}, V_{s_i})\|_{D_{s_{i-1}, s_i}}.$$

This minimum is attained for $t = (s_{i-1}V_{s_{i-1}} + s_iV_{s_i})/(V_{s_{i-1}} + V_{s_i})$.

Now consider a sequence of discretized versions $\hat{\mathbf{V}}^{(d)}$ of an SGPP \mathbf{V} with grid \mathcal{G}_d , where the fineness of \mathcal{G}_d converges to zero. Repeating the arguments in the proof of Theorem 4.5 yields the following result.

Theorem 4.5. *The sequence of processes $\hat{\mathbf{V}}^{(d)}$, $d \in \mathbb{N}$, converges uniformly to \mathbf{V} , i. e. $\|\hat{\mathbf{V}}^{(d)} - \mathbf{V}\|_\infty \rightarrow_{d \rightarrow \infty} 0$.*

4.2. The Mean Squared Error of the Discretized Version. The aim of this section is the calculation of the mean squared error of the predictor \hat{V}_t of V_t . We obtain again some kind of pointwise convergence in mean square of a sequence of discretized versions with increasing fineness to the initial SGPP. Nevertheless, there is a difference to the considerations in the previous section. In contrast to the case of max-stable distributions, we typically only know the distribution of a GPD rv in the upper tail. Note that the function $W(\mathbf{x}) := 1 - \|\mathbf{x}\|_D$, $\mathbf{x} \leq 0$, $\|\mathbf{x}\|_D \leq 1$, does not define a multivariate df in general, see cf. Falk et al. (2010). This fact forces us to consider conditional expectations in this section.

In the bivariate case, however, W defines a df, and we can assume that a GPD has this representation on the whole domain. The next Lemma is on some conditional moments of bivariate standard GPD rv in general.

Lemma 4.6. *Let (U, V) be a standard GPD rv, i. e. there exists some D -norm $\|\cdot\|_D$ such that $P(U \leq u, V \leq v) = 1 - \|(u, v)\|_D$, $u, v \leq 0$, $\|(u, v)\|_D \leq 1$. Then we have for all such u, v*

(i)

$$P(U > u, V > v) = \|(u, v)\|_1 - \|(u, v)\|_D,$$

and, in case of $\|\cdot\|_D \neq \|\cdot\|_1$,

(ii)

$$\begin{aligned} & E(U^2 | U > u, V > v) \\ &= -\frac{\frac{2}{3}u^3 + u^2(u + \|(u, v)\|_D) + v^3 \int_0^{u/v} \int_0^{u/v} \|(\max(s, t), 1)\|_D \, ds \, dt}{\|(u, v)\|_1 - \|(u, v)\|_D}, \end{aligned}$$

(iii)

$$E(UV|U > u, V > v) = - \frac{\int_u^0 \int_u^0 \|(s, t)\|_D \, ds \, dt + v^3 \int_0^{u/v} \|(r, 1)\|_D \, dr}{\|(u, v)\|_1 - \|(u, v)\|_D} - \frac{u^3 \int_0^{v/u} \|(1, r)\|_D \, dr + uv \|(u, v)\|_D}{\|(u, v)\|_1 - \|(u, v)\|_D}.$$

Note that the case $\|\cdot\|_D = \|\cdot\|_1$ has to be treated with caution. It represents the case of uniform distribution on the line $\{(x, y) : x, y \leq 0, x + y = -1\}$, which means that no observations fall in any rectangular $[u, 0] \times [v, 0]$, $u + v \geq -1$, cf. Falk et al. (2010).

Proof of Lemma 4.6. (i) We have

$$\begin{aligned} P(U > u, V > v) &= 1 - P(U \leq u) - P(V \leq v) + P(U \leq u, V \leq v) \\ &= 1 - (1 + u) - (1 + v) + 1 - \|(u, v)\|_D \\ &= \|(u, v)\|_1 - \|(u, v)\|_D. \end{aligned}$$

(ii) We obtain by Fubini's theorem and elementary computations

$$\begin{aligned} &E(1_{\{U > u, V > v\}} U^2) \\ &= \int_{[u, 0] \times [v, 0]} x^2 (P * (U, V))(d(x, y)) \\ &= \int_{[u, 0] \times [v, 0]} \left(\int_u^0 \int_u^0 1_{[x, 0]}(s) \cdot 1_{[x, 0]}(t) \, ds \, dt \right) (P * (U, V))(d(x, y)) \\ &= \int_u^0 \int_u^0 \left(\int_{[u, 0] \times [v, 0]} 1_{[x, 0]}(s) \cdot 1_{[x, 0]}(t) (P * (U, V))(d(x, y)) \right) ds \, dt \\ &= \int_u^0 \int_u^0 \left(\int_{[u, \min(s, t)] \times [v, 0]} (P * (U, V))(d(x, y)) \right) ds \, dt \\ &= \int_u^0 \int_u^0 P(U \in [u, \min(s, t)], V \in [v, 0]) \, ds \, dt \\ &= \int_u^0 \int_u^0 P(U \leq \min(s, t)) \, ds \, dt - \int_u^0 \int_u^0 P(U \leq u) \, ds \, dt \\ &\quad + \int_u^0 \int_u^0 P(U \leq u, V \leq v) \, ds \, dt - \int_u^0 \int_u^0 P(U \leq \min(s, t), V \leq v) \, ds \, dt \\ &= \int_u^0 \int_u^0 1 - \|(s, t)\|_\infty \, ds \, dt - u^2 (1 + u) \\ &\quad + u^2 (1 - \|(u, v)\|_D) - \int_u^0 \int_u^0 1 - \|(\min(s, t), v)\|_D \, ds \, dt \\ &= -\frac{2}{3} u^3 - u^2 (u + \|(u, v)\|_D) + \int_u^0 \int_u^0 \|(\min(s, t), v)\|_D \, ds \, dt \end{aligned}$$

$$= -\frac{2}{3}u^3 - u^2(u + \|(u, v)\|_D) - v^3 \int_0^{u/v} \int_0^{u/v} \|(\max(r_1, r_2), 1)\|_D \, dr_1 \, dr_2,$$

where we substitute $r_1 = s/v$ and $r_2 = t/v$ in the last equality. Together with assertion (i), the statement is proven.

(iii) Similar arguments as in the proof of (ii) yield

$$\begin{aligned} E(1_{\{U>u, V>v\}}UV) &= \int_v^0 \int_u^0 P(U \in [u, s], V \in [v, t]) \, ds \, dt \\ &= \int_v^0 \int_u^0 P(U \leq s, V \leq t) \, ds \, dt - \int_v^0 \int_u^0 P(U \leq s, V \leq v) \, ds \, dt \\ &\quad - \int_v^0 \int_u^0 P(U \leq u, V \leq t) \, ds \, dt + \int_v^0 \int_u^0 P(U \leq u, V \leq v) \, ds \, dt \\ &= - \int_v^0 \int_u^0 \|(s, t)\|_D \, ds \, dt + \int_u^0 \int_v^0 \|(s, v)\|_D \, dt \, ds \\ &\quad + \int_v^0 \int_u^0 \|(u, t)\|_D \, ds \, dt - \int_v^0 \int_u^0 \|(u, v)\|_D \, ds \, dt \\ &= - \int_v^0 \int_u^0 \|(s, t)\|_D \, ds \, dt - v \int_u^0 \|(s, v)\|_D \, ds \\ &\quad - u \int_v^0 \|(u, t)\|_D \, dt - uv \|(u, v)\|_D \\ &= - \int_v^0 \int_u^0 \|(s, t)\|_D \, ds \, dt - v^3 \int_0^{u/v} \|(r, 1)\|_D \, dr \\ &\quad - u^3 \int_0^{v/u} \|(1, r)\|_D \, dr - uv \|(u, v)\|_D. \end{aligned}$$

□

EXAMPLE 4.7. *In case of total dependence of U and V (i. e. $\|\cdot\|_D = \|\cdot\|_\infty$) and $u = v =: c$, the formulas in Lemma 4.6 (ii) and (iii) become*

$$E(U^2|U > u, V > v) = -\frac{\frac{2}{3}c^3 + c^2(c - c) + c^3}{-c} = \frac{5}{3}c^2$$

and

$$E(UV|U > c, V > c) = -\frac{-\int_c^0 \int_c^0 \min(s, t) \, ds \, dt + c^3 + c^3 - c^3}{-c} = \frac{5}{3}c^2.$$

We now return to the discretized Version $\hat{\mathbf{V}}$ of an SGPP \mathbf{V} . Again, we have to show that for every $t \in [0, 1]$ the rv (V_t, \hat{V}_t) follows a standard GPD. The exact same arguments as in Lemma 3.8 provide the bivariate df of this rv, at least in the upper tail.

Lemma 4.8. *Let $\mathbf{V} = (V_t)_{t \in [0,1]}$ be an SGPP with generator $\mathbf{Z} = (Z_t)_{t \in [0,1]}$. Denote by $\hat{\mathbf{V}} = (\hat{V}_t)_{t \in [0,1]}$ its discretized version with grid $\{s_0, \dots, s_d\}$ and generator $\hat{\mathbf{Z}} = (\hat{Z}_t)_{t \in [0,1]}$. Then (V_t, \hat{V}_t) defines a bivariate standard GPD rv for every $t \in [0, 1]$. Its df is given by*

$$\begin{aligned} P(V_t \leq x, \hat{V}_t \leq y) &= 1 - \|(x, y)\|_{D_t} \\ &= 1 - \left\| \left(x, g_{i-1}(t)y, g_i(t)y \right) \right\|_{D_{t, i-1, i}}, \quad t \in [s_{i-1}, s_i], \quad i = 1, \dots, d, \end{aligned}$$

for x, y close enough to zero, where $\|\cdot\|_{D_t}$ is the D -norm generated by (Z_t, \hat{Z}_t) and $\|\cdot\|_{D_{t, i-1, i}}$ is the D -norm generated by $(Z_t, Z_{s_{i-1}}, Z_{s_i})$.

We close this section with the calculation of the mean squared error of \hat{V}_t , under the condition that V_t and \hat{V}_t attain values that are close enough to zero, such that we have a representation of the df of (V_t, \hat{V}_t) available in this area. Again, this means squared error converges to zero.

Suppose \mathbf{V} is an SGPP with generator \mathbf{Z} and choose a sequence of grids \mathcal{G}_d of the interval $[0, 1]$ with fineness converging to zero as d increases. Denote by $\hat{\mathbf{V}}^{(d)}$, $d \in \mathbb{N}$, the sequence of discretized versions of \mathbf{V} with grid \mathcal{G}_d , and by $\hat{\mathbf{Z}}^{(d)}$, $d \in \mathbb{N}$, their generators. Denote further by $\|\cdot\|_{D_t^{(d)}}$ the D -norm generated by $(Z_t, \hat{Z}_t^{(d)})$, $t \in [0, 1]$, $d \in \mathbb{N}$.

Theorem 4.9. *Let \mathbf{V} and $\hat{\mathbf{V}}^{(d)}$, $d \in \mathbb{N}$, be as above. Suppose $\|\cdot\|_{D_t^{(d)}} \neq \|\cdot\|_1$, $d \in \mathbb{N}$. Then we have for c close enough to zero*

$$E \left(\left(V_t - \hat{V}_t^{(d)} \right)^2 \middle| V_t > c, \hat{V}_t^{(d)} > c \right) \rightarrow_{d \rightarrow \infty} 0.$$

Proof. According to Lemma 4.8, the rv $(V_t, \hat{V}_t^{(d)})$, $d \in \mathbb{N}$, is a standard GPD rv with pertaining D -norm $\|\cdot\|_{D_t^{(d)}}$. We have already shown in the proof of Theorem 3.9 that $\|\cdot\|_{D_t^{(d)}} \rightarrow_{d \rightarrow \infty} \|\cdot\|_\infty$ pointwise. Substituting $\|\cdot\|_{D_t}$ by $\|\cdot\|_1$ in the numerators of Lemma 4.6 (ii) and (iii) leads to finite integrals exclusively, which is why we can apply the dominated convergence theorem in each of these integrals. Therefore, we obtain by the calculations in Example 4.7 for all $t \in [0, 1]$

$$\begin{aligned} &E \left(\left(V_t - \hat{V}_t^{(d)} \right)^2 \middle| V_t > c, \hat{V}_t^{(d)} > c \right) \\ &= E \left(V_t^2 \middle| V_t > c, \hat{V}_t^{(d)} > c \right) - 2E \left(V_t \hat{V}_t^{(d)} \middle| V_t > c, \hat{V}_t^{(d)} > c \right) \\ &\quad + E \left(\left(\hat{V}_t^{(d)} \right)^2 \middle| V_t > c, \hat{V}_t^{(d)} > c \right) \\ &\rightarrow_{d \rightarrow \infty} \frac{10}{3}c^2 - \frac{10}{3}c^2 = 0. \end{aligned}$$

□

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